



Branes in hearts with perverse sheaves

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Abstract Various topological properties of D-branes in the type-IIA theory are captured by the topologically twisted model, treating D-branes as objects in the bounded derived category of coherent sheaves on the compact part of the target space. The set of basic D-branes wrapped on the homology cycles of the compact space are taken to reside in the set of t-structures of the derived category of coherent sheaves on the space at any point in the Kähler moduli space. The stability data entails specifying a t-structure along with a grade for sorting the branes. Considering an example of a degenerate Calabi-Yau space, obtained via geometric engineering, that retains but a projective curve as the sole compact part, we identify the regions in the Kähler moduli space of the curve that pertain to the different t-structures in the bounded derived category of coherent sheaves on the curve corresponding to the different phases of the topological branes.

Keywords D-brane, topological B-model, derived category, homological algebra

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A physically inclined overture

One of the objects in the spectrum of string theory space, extended beyond a single dimension have received appellation, D-branes [1]. These objects have played a

pivotal role in the exploration of various non-perturbative aspects of string theory and its sundry consequences [2–7]. Depending on the type or version of string theory one considers and the point of view, D-branes are described by different types of data. From a conformal field theoretic point of view, for example, D-branes correspond to Dirichlet boundary conditions imposed on the end-points of open strings in a classical picture, lending the suffix *D* to the nomenclature. In a low-energy target-space description, on the other hand, D-branes are described as certain extended solitonic solutions of the effective gravitational theories, carrying the Ramond-Ramond fields in supersymmetric instances. A *Dp*-brane is an object extending in *p* space directions, the part *brane* having been extracted from a two-dimensional *membrane* of which it is imagined to be a dimensionally generalized form. A related description of a *Dp*-brane is in terms of the Dirac-Born-Infeld theory on its $(p+1)$ -dimensional world-volume, which in appropriate limits of coupling reduces to a gauge theory. The geometric character of the gravitational or gauge theories has instigated endeavors to unearth a canonical definition of D-branes in string theory in terms of purely geometric data. This has been achieved for certain restricted classes

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of D-branes in certain classes of string theories, no ordinary a feat, considering the intricacies involved, as we shall remark in the sequel. In this article we discuss the definition of BPS D-branes in the topological B-model as certain stable objects in the bounded derived category of coherent sheaves on the target-space. Before plunging into this, let us briefly discuss some of the physical properties of D-branes and examine the inadequacy of the possible “intuitive” geometric definitions within the context of the theories of which allusion has been made above.

Let us begin with the world-sheet description of D-branes. The importance of this description lies in its ubiquity, as the validity of any geometric definition is to be tested against expectations from a conformal field theoretic description [8, 9]. The non-linear sigma-model action describing the world-sheet of a string is [10, 11]

$$S = \int_S d^2\sigma (g_{MN} + b_{MN}) \eta^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N, \quad (1.1)$$

where $\alpha, \beta = 0, 1$, σ^0 and σ^1 denote, respectively, the temporal and the spatial coordinates of the world-sheet of the string and η denotes the flat metric on the world-sheet. The fields X^M are interpreted as the coordinates of the target-space, with M, N ranging over the dimen-

sions of the target-space, ten for superstrings and twenty-six for bosonic strings, with a fermionic piece for the latter, which will not feature in our discussion at this point. The coupling parameters of the sigma-model constitute a symmetric matrix g_{MN} , interpreted as the metric on the target-space and an anti-symmetric one, b_{MN} , known as the Kalb-Ramond field, giving rise to torsion in the target-space. The integration is over the area of the world-sheet, S . The Euler-Lagrange equations ensuing from the action is the two-dimensional Laplace's equation for each field X^M ,

$$\eta_{\alpha\beta} \partial_\alpha \partial_\beta X^M = 0$$

For open strings, that is strings extending between $\sigma^1 = 0$ and $\sigma^1 = \pi$, we can solve the Laplace's equation by choosing either a Neumann or a Dirichlet boundary condition on the edges. Moreover, since the fields X^M are independent of each other, we can choose different boundary conditions for the different X^M . While choosing the Neumann condition for some X^M leaves the end-point of the open string dangling in that direction of the target-space, the Dirichlet condition for some of the fields requires fixing the edge at a certain point in the respective direction as shown in Figure 1(a), resulting into a breakdown of Poincaré invariance in the target-space, the *space-time*

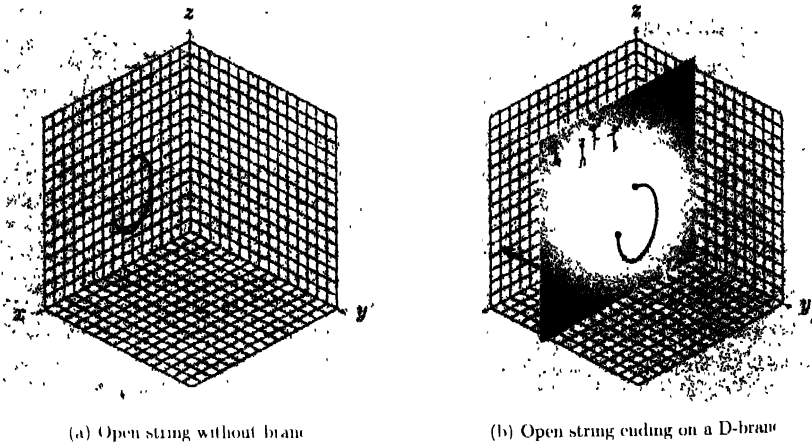


Figure 1: Open strings with and without D-branes

An awkward predicament as such may be avoided by adjoining that the string is not stuck at a special point in space-time, but on an *object* that perambulates the target-space, as illustrated in Figure 1(b). This object is a *brane*. If we consider an open string with the end $\sigma^1 = 0$ a D-brane, then the dimension p of the D p -brane equals the number of X 's on which Dirichlet condition is imposed $\neq 0$. Thus, a superstring theory may have D-branes of maximal dimension nine, which pervades all of space-time. If we consider a D-brane in the target-space, then, it appears that a D-brane can be considered as a part of the space-time, a *subspace*.

Gauge fields provide further embellishment to this picture of D-branes. An end-point of an open string may support gauge degrees of freedom, known as Chan-Paton factors. In the world-sheet description, this is incorporated by adjoining a term [6, 10, 11]

$$S_{\text{gauge}} = \text{Tr} \int_{\partial S} A_M \partial_0 X^M \quad (1.3)$$

where the action S in (1.1). Here A_M denotes a gauge field and the integration is over the boundary of the world-sheet ∂S , the trace being over the gauge indices. The corresponding effective field theory yields the world-volume theory of D-branes as the Dirac-Born-Infeld action [11, 12],

$$S_{\text{BI}} = \text{Tr} \int_V \sqrt{\det(G + B + F)}, \quad (1.4)$$

where G is the metric induced from the space-time on the world-volume of the brane, B an anti-symmetric field induced from the Kalb-Ramond field and F denotes the $U(1)$ field-strength corresponding to A . The integration is over the world-volume of the brane, V . The first non-trivial leading order term in the expansion of the square-root yields a Yang-Mills theory on the world-volume of the brane. Thus, supplementing the earlier description, it appears that a D p -brane may be described as a vector bundle on a p -dimensional subspace of the target-space.

Let us now proceed to consider compactification of string theory and dualities. From now on we restrict ourselves to considering superstring theory only. First, let us consider a simple case, a supersymmetric BPS D4-brane in the type-IIA theory compactified on a two-dimensional torus, T^2 . Let us assume that the brane lies along the directions of X^6, X^7, X^8 and X^9 of the target-space and the directions along X^4, X^5 . If we now perform two T-duality transformations along the two directions of the torus separately, then since T-duality exchanges Neumann with Dirichlet boundary condition, the brane extends along the

torus, turning into a D6-brane. However, T-duality is assumed to be a symmetry of string theory and two such consecutive operations are supposed to leave the type-IIA theory unaltered. Hence the two theories must be identified along with their spectra. In other words, in the type-IIA theory on a torus, the transmogrification of a D4-brane into a D6-brane must be allowed. This example exhibits that duality transformations do not respect dimensionality of branes, dimensions of branes change as we change the parameters of compactification, the *moduli*, of the target-space, perhaps by a T-duality transformation. Thus the description of D-branes as (bundles on) some subspace of fixed dimension of the target is inadequate.

Next, let us consider the example of the type-IIA theory on the target-space $\mathbf{R}^{1,3} \times \mathcal{M}$, where \mathcal{M} is a six-dimensional Calabi-Yau manifold. Let us consider a BPS D0-brane in this four-dimensional theory. A BPS brane is a supersymmetric solution of the corresponding supergravity theory obeying the BPS conditions. Many of the topological properties of these are described by the topologically twisted B-model, the branes being B-branes. Since the BPS branes in the type-IIA theory are even-dimensional, and let us note that this specification is free from the caveat alluded to above, the D0-brane in four-dimensions may arise from a D6-brane wrapping the six-dimensional manifold \mathcal{M} or a D p -brane wrapping a supersymmetric p -cycle of the homology of \mathcal{M} , for $p = 4, 2, 0$. As discussed above, in attempting to describe the D0-brane as a vector bundle, it can be viewed as either a vector bundle above \mathcal{M} or one of these cycles or their combination. However, as we change the size of the Calabi-Yau manifold, by the available Kähler structure moduli, cycles in the homology transmute among each other. For example, a four-cycle may go into a combination of a four-cycle and a two-cycle or vice versa. Thus, bundles on cycles is not a very useful specification; we need to describe the bundles in terms of structures on the six-dimensional Calabi-Yau manifold itself. This can be achieved by extending a bundle on a lower dimensional cycle to the whole of \mathcal{M} by a zero section — so viewed over \mathcal{M} the rank of the bundle jumps. We are thus led to consider sheaves on the manifold \mathcal{M} . The collection of branes in such a scenario leads us to considering a category of sheaves on \mathcal{M} , the open string stretched between branes providing the morphisms in the category. Transmutation of cycles can be thought of as formation or decay of bound states of branes, when branes are taken to be these sheaves.

In the collection of branes we need to include anti-branes too. A brane and its corresponding anti-brane dif-

fer by a change of sign of the charge they carry. Considering branes and anti-branes within the same schemata forces us to consider not only sheaves but complexes of sheaves. We shall discuss more on this in the next section.

Now, as we deform \mathcal{M} , there are points in the Kähler moduli space of \mathcal{M} where a brane decays or branes form bound states, as mentioned above. Right at this point, called the point of *marginal stability*, [13–19] the brane or the collection of its decay products or components can not be distinguished. Incorporation of this indistinguishability necessitates the identification of a sheaf and its resolutions, ushering in the appearance of derived functors into the arena. Consistency with the conjectured mirror symmetry, which is envisaged as a geometric realization of T-duality, at least in part, requires these sheaves to be coherent. We are thus led to consider the derived category of coherent sheaves [20] on \mathcal{M} . In addition, in the physically interesting cases a BPS brane decays into a finite number of products or bound states of only a finite number of branes are considered. This restricts us to consider finite complexes of sheaves. We thus arrive at the bounded derived category of coherent sheaves on \mathcal{M} . Finally, BPS branes are stable in a certain sense, unstable configurations decay to a stable one. We shall expatiate on the stability criterion in the next section. In the case of type IIB theory arguments similar to the ones above are valid. But in that case one needs to consider odd-dimensional cycles and the description of the category of branes is more complicated.

To summarize, we have motivated the description of BPS D-branes in the type-IIA theory as stable objects in the derived category of sheaves on a compact part of the target-space. We now proceed to discuss the concepts further in the next section.

2 Viewing homologically

“A derived category is . . .
when you take complexes seriously” [21].

In this section we recall [21] some features of derived categories relevant for our discussion of physical situations in the following sections [14–17, 20–27].

Let \mathcal{A} denote an *Abelian category*, that is, a category with kernels and co-kernels of morphisms defined within. Given an Abelian category \mathcal{A} , the corresponding *derived category* $D(\mathcal{A})$ consists of complexes A^\bullet of objects of \mathcal{A} ,

viz

$$A^\bullet = \cdots \xrightarrow{d_{n-2}} A_{n-1} \xrightarrow{d_{n-1}} A_n \xrightarrow{d_n} A_{n+1} \xrightarrow{d_{n+1}} \cdots$$

$$d_n \circ d_{n+1} = 0, \quad A_i \in \mathcal{A},$$

upto identification by quasi-isomorphisms. A *quasi-isomorphism* is a morphism of complexes inducing an isomorphism on cohomology. A derived category is *bounded* if the complexes have only a finite number of non-vanishing elements. A bounded derived category corresponding to \mathcal{A} is denoted $D^b(\mathcal{A})$. If A^\bullet represents an object in $D^b(\mathcal{A})$ then it is quasi-isomorphic to a certain complex I^\bullet of injectives bounded below,

$$0 \longrightarrow A^\bullet \longrightarrow I^\bullet.$$

Example 2.1 Let $A^\bullet = \cdots \longrightarrow \underline{A} \longrightarrow \cdots$ be a complex concentrated in degree zero, the underlined entry. A quasi-isomorphism $A^\bullet \longrightarrow I^\bullet$ is an injective resolution A , that is, an exact sequence,

$$0 \longrightarrow A \xrightarrow{\epsilon} I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} \cdots$$

Then the cohomologies are isomorphic, $H^*(I^\bullet) \sim H^*(A)$ where one defines $H^0(I^\bullet) = \ker d_0 \simeq \operatorname{Im} \epsilon \simeq H^0(A)$.

Two injective resolutions of A are isomorphic to A in the derived category and thus to each other, imply that they are homotopy equivalent. These statements continue to be valid, mutatis mutandis, with A replaced by a complex A^\bullet in the above example. Physically, a D-brane corresponds to a complex A^\bullet in $D^b(\mathcal{A})$ for some Abelian category \mathcal{A} , to be specified later.

2.1 Shifts and triangles of complexes

Among the structures a derived category is endowed with are *translations* and *triangles*. A translation is a functors defined as a shift in the left of the entries of a complex, namely,

$$[n] : A^\bullet \longrightarrow A^\bullet[n], \\ A[n]_m = A_{m+n}$$

Example 2.2 Defining the *mapping cone* of a morphism $A^\bullet \xrightarrow{f} B^\bullet$ between two complexes as the complex $M(f) = A^\bullet[1] \oplus B^\bullet$ with appropriately shifted morphisms we naturally obtain the complex

$$A^\bullet \xrightarrow{f} B^\bullet \longrightarrow M(f) \longrightarrow A^\bullet[1],$$

where the last term is the complex A^\bullet shifted by unit degree.

physically, if a D-brane corresponds to a complex A^\bullet , then the shifted complex $A^\bullet[1]$, and any of its cousins shifted by odd degree thereof, corresponds to the anti-D-brane. The justification for this interpretation is derived from the fact that, if we define the Chern character map from (\mathcal{A}, A) to cohomology, then $\text{ch}(A^\bullet[n]) = (-1)^n \text{ch}(A^\bullet)$, for an integer n and the charge of a D-brane A^\bullet is taken to be proportional to $\text{ch}(A^\bullet)$. Moreover, $M(f)$ is taken to represent the marginal deformation of the configuration consisting of the two D-branes corresponding to A^\bullet and B^\bullet .

Triangles in the derived category, on the other hand, have counterparts of the exact sequences in an Abelian category.

Example 2.3 Let us consider a short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in the Abelian category \mathcal{A} . Let us form a complex $C^\bullet = 0 \rightarrow A \rightarrow B \rightarrow 0$. Then this complex is isomorphic to C in the derived category, $C^\bullet \simeq C$, as discussed above. For this realization C^\bullet of C we have the triangle $C^\bullet \rightarrow A[1] \rightarrow A[1]$, namely,

$$\begin{array}{ccccccc} C^\bullet & & 0 & \rightarrow & A & \rightarrow & B \rightarrow 0 \\ \downarrow & & & & \downarrow & & \downarrow \\ A[1] & \rightarrow & 0 & \rightarrow & A & \rightarrow & B \rightarrow 0 \end{array}$$

which completes the triangle $A \rightarrow B \rightarrow C \rightarrow A[1]$ as written in a more picturesque form as

$$\begin{array}{ccc} & C & \\ [1] \swarrow & & \searrow \\ A & \xrightarrow{\quad} & B \end{array}$$

The recurrence of triangles makes the derived category into a triangulated category. Again, all the statements above are valid when the objects are replaced by complexes. \square

A triangle is called *distinguished* if it is quasi-isomorphic to (2.4). While D-branes are represented by complexes, a oriented open string stretched between two D-branes A^\bullet and B^\bullet is taken to be represented by a distinguished triangle. The triangle $A^\bullet \xrightarrow{1} A^\bullet \rightarrow 0 \rightarrow A[1]$ is considered distinguished by hypothesis in a derived category. In the light of the interpretation of a shifted complex as an anti-brane, this triangle is taken to represent the annihilation of a brane-anti-brane pair. Triangles in a derived category lead to what is known as the *fearful symmetry*. In the short exact sequence

$0 \rightarrow A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow 0$ we can think of B^\bullet as being “made up of” A^\bullet and C^\bullet , with C^\bullet over A^\bullet . But in the derived category, corresponding to a triangle

$$A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow A^\bullet[1], \quad (2.5)$$

there exists a homotopy-equivalent triangle

$$B^\bullet \rightarrow C^\bullet \rightarrow A^\bullet[1] \rightarrow B^\bullet[1] \quad (2.6)$$

Extending the above interpretation then implies that C^\bullet is made up of objects B^\bullet and $A^\bullet[1]$, with $A^\bullet[1]$ over B^\bullet . In particular, the notion of *subobjects* ceases to exist. Physically, it turns out to be convenient to interpret C^\bullet as a potential bound state of A^\bullet and B^\bullet , corresponding to the triangle (2.5). From the triangle (2.6), however, A^\bullet , shifted by unit degree, appears as a bound state of B^\bullet and C^\bullet . Similarly, even B^\bullet can be thought of as a potential bound state of A^\bullet and $C^\bullet[-1]$, by reading the triangle (2.5) toward left. This upsets any notion of order in the derived category. Our goal, on the other hand, is to study *stable* BPS branes. Any notion of stability calls for an order — between an object and its components. At this point let us only remark that while such an order is not defined for the objects in a derived category, the Abelian category does allow for an order. Moreover, it is apparent from the discussion above that the same triangulated category can be represented as the derived category of different Abelian categories. These will be relevant in ascertaining stability of branes, while still defining them to be objects in a derived category.

Finally, let us discuss the *octahedral axiom*, which can be used in realizing an isomorphism between mapping cones of two composed maps. Given two distinguished triangles

$$\begin{array}{ccccccc} A^\bullet & \xrightarrow{f} & B^\bullet & \xrightarrow{g} & C^\bullet & \xrightarrow{h} & A^\bullet[1], \\ B^\bullet & \xrightarrow{\beta} & E^\bullet & \xrightarrow{\gamma} & F^\bullet & \xrightarrow{\delta} & B^\bullet[1], \end{array} \quad (2.7)$$

there exists a complex M^\bullet and two distinguished triangles,

$$\begin{array}{ccccccc} A^\bullet & \xrightarrow{\beta \circ f} & E^\bullet & \xrightarrow{\tau} & M^\bullet & \xrightarrow{\omega} & A^\bullet[1], \\ C^\bullet & \xrightarrow{\sigma} & M^\bullet & \xrightarrow{\nu} & F^\bullet & \xrightarrow{g[1] \circ \delta} & C^\bullet[1], \end{array} \quad (2.8)$$

such that the diagram

$$\begin{array}{ccccccc}
 A^* & \xrightarrow{f} & B^* & \xrightarrow{g} & C^* & \xrightarrow{h} & A^*[1] \\
 \downarrow 1 & & \downarrow \mu & & \downarrow \sigma & & \downarrow 1 \\
 A^* & \xrightarrow{\beta \circ f} & E^* & \xrightarrow{\tau} & M^* & \xrightarrow{\omega} & A^*[1] \\
 \downarrow f & & \downarrow 1 & & \downarrow \nu & & \downarrow f[1] \\
 B^* & \xrightarrow{\beta} & E^* & \xrightarrow{\gamma} & F^* & \xrightarrow{\delta} & B^*[1] \\
 \downarrow g & & \downarrow \tau & & \downarrow 1 & & \downarrow g[1] \\
 C^* & \xrightarrow{\sigma} & M^* & \xrightarrow{\nu} & F^* & \xrightarrow{g[1] \circ \delta} & C^*[1]
 \end{array} \quad (2.9)$$

commutes. Even though it does not appear obvious, the above diagram can indeed be cast into an octahedral form. However, drawing it in that fashion does not make reading of the diagram any easier. We shall discuss an application of this in §2.4.

2.2 A glimpse of the heart

The formalism of t-structures is a means to vivisect a triangulated category and identify the various Abelian subcategories in it. Let us recall some related definitions.

For an Abelian category \mathcal{A} and its derived category $D(\mathcal{A})$, let $D^{\leq n}$ denote the full subcategory, that is, one whose morphisms coincide with the morphisms of the parent category, of $D(\mathcal{A})$ formed by complexes A^* with cohomology only beyond n , that is, $H^i(A^*) = 0$ for $i < n$. Similarly, let $D^{\geq n}$ denote the full subcategory of $D(\mathcal{A})$ of complexes A^* with cohomology only below n , that is $H^i(A^*) = 0$ for $i > n$. A t-structure on a triangulated category is a pair $(D^{\leq 0}, D^{\geq 0})$ of strictly full subcategories, satisfying the following conditions

1. $D^{\leq 0} \subset D^{\leq 1}$ and $D^{\geq 1} \subset D^{\geq 0}$,
2. $\text{Hom}^0(D^{\leq 0}, D^{\geq 1}) = 0$,
3. For each object K of the triangulated category, there exists a distinguished triangle

$$\begin{array}{ccc}
 & K_{\geq 1} & \\
 \nearrow & & \searrow \\
 K & \xleftarrow{\quad} & K_{\leq 0}
 \end{array}$$

Further, a t-structure is called *bounded* if each K in the triangulated category is contained in $D^{\geq m} \cap D^{\leq n}$ for some integers m and n . The intersection $D^{\geq 0} \cap D^{\leq 0}$ coincides with \mathcal{A} , now called the *heart* of the t-structure. A bounded t-structure is completely described by its heart. We use bounded t-structures only and describe them using the

heart. Physically, the heart of a t-structure provides the basic brane configurations with which we can construct the others. Let us now turn to discussing the stability criterion for objects in a derived category.

2.3 Stable branes

The definition of a stability criterion in a triangulated category, of which a derived category is an example, owes its origin largely to the physics of BPS branes [17]. Mathematically, it is a generalization of the μ -stability condition in the Abelian category of coherent sheaves on a non-singular projective curve, defined using a Harder-Narasimhan filtration [25–28]. As stressed above, a stability criterion requires sorting the objects, which implicitly calls for the notion of an index. While a triangulated category obfuscates such a notion, an Abelian category is amenable to it. Thus, to impose an order on a triangulated category one first identifies an Abelian category within the triangulated category using t-structures and then orders the objects in the Abelian category. The latter is achieved through defining a *centered slope function* on an Abelian category \mathcal{A} as a group homomorphism $Z: K(\mathcal{A}) \rightarrow \mathbb{C}$ from the Grothendieck group $K(\mathcal{A})$ of \mathcal{A} , such that for non-zero objects E of \mathcal{A} the complex number $Z(E)$ lies in the strict upper-half plane – the upper half of the complex plane sans the real axis. The *phase* of a non-zero object E of \mathcal{A} is defined in terms of the slope function

$$\varphi(E) = -\frac{1}{\pi} \arg Z(E) \in [-1, 0). \quad (2)$$

The non-zero objects of the Abelian category \mathcal{A} can be ordered by their phases, φ . Indeed, a non-zero object of \mathcal{A} is said to be *semistable* if $\varphi(A) \leq \varphi(E)$ for every non-zero subobject A of E . A slope function Z defined on \mathcal{A} is said to have the *Harder-Narasimhan property* if every non-zero object of \mathcal{A} has a Harder-Narasimhan filtration. That is, for each object E of \mathcal{A} , there is a finite chain of subobjects

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E,$$

such that the quotients $F_i = E_i/E_{i-1}$ are semistable objects of \mathcal{A} with

$$\varphi(F_1) > \varphi(F_2) > \cdots > \varphi(F_n). \quad 2$$

We have remarked above that the heart of the t-structure is an Abelian category. Specifying a stability condition on a triangulated category is equivalent to giving a bounded t-structure on it along with a centered slope function on the heart of the t-structure with the Harder-Narasimhan property [25]. This is called the *II-stability*, II being the traditional symbol for periods of homology cycles.

The central charge

portrayal of stable D-branes as objects in the derived category of coherent sheaves on the target space thus necessitates specifying a centered slope function with the Harder-Narasimhan property. For B-type branes such a function is furnished by the central charge of the superconformal algebra of BPS branes. Indeed, it was this central charge that motivated the mathematical definition of the slope function. We denote the central charge of a brane by Z too. The phase ϕ of Z , called the *grade*, defines the centered slope function with the Harder-Narasimhan property determining the stability of branes. Let us consider the type-IIA theory compactified on a Calabi-Yau manifold, \mathcal{M} . Let E and F be two D-branes in the derived category $D^b(\text{Coh } \mathcal{M})$ and let $C = M(f : E \rightarrow F)$ be the mapping cone mentioned earlier, which corresponds to a singular deformation of the branes E and F , a potential-bound state. If it is marginally stable, then we have,

$$Z(F) - Z(E), \text{ implying} \quad \phi(C) = \phi(F) = \phi(E) + 1 \quad (2.12)$$

grade ϕ depends on the Kähler parameter of \mathcal{M} thus given two branes E and F in the derived category (2.12) determines the set of points in the Kähler moduli space where the bound state C is marginally stable [19]. This set is called the *line of marginal stability*. Given a point on this line, three cases arise as we move off this point in the Kähler moduli space, namely, $\Delta\phi = \phi(F) - \phi(E) - 1 \gtrless 0$. If the expression is negative, an open string joining the branes becomes tachyonic (C becomes a bound state). If $\Delta\phi$ is positive, on the other hand, then C is unstable against decay to E and F . If the expression continues to be equal to zero, then the bound state remains settled into a stasis on the line of marginal stability. This is illustrated in the Figure 2(a). As an analysis of II-stability involves two steps. First, the marginal stability line is to be obtained. Then the stability of branes is to be checked as a second step as one moves off this line. If it all seems copacetic so far, let us consider the complications arise in ascertaining the stability of branes from stability issues of the components. For example, even if the expression $\phi(F) - \phi(E) - 1$ is negative,

it is not possible to aver that C is stable. Indeed, it is not improbable that the derived category contains another triangle with C at a vertex. C may then be a bound state of certain other branes, $E' \rightarrow F'$. In other words, while C is stable with respect to decaying into E and F , it may have other decay channels. If, moreover, the expression is positive, we can not conclude with certitude that C is unstable, since the components E and F may themselves be unstable and then the bound state may be perdurable, being energetically favored. A quandary as such can be settled if we have means to compare two decay channels, or, in other words, we can decide whether two distinguished triangles are isomorphic or not. As discussed before, this can be done using the octahedral axiom. We now proceed to discuss the issue in some detail as an illustration of the intricate internal consistency of the definitions [29].

Let us consider an object C which is known to be stable at a given point in the moduli space p , marked in Figure 2(b). Let a decay channel of C be $C \rightarrow A + B$ across the line L_C with $\phi(B) = \phi(A) + 1$ on the line. Thus, if A and B are both stable, then L_C is a line of marginal stability for this decay, with C stable in the region $\mathfrak{S}_2 \cup \mathfrak{S}_4$. Similarly, let us suppose that there is another possible decay $B \rightarrow E + F$ across the line L_B , B being stable in the region $\mathfrak{S}_1 \cup \mathfrak{S}_3$. Let us now consider two different paths P_1 and P_2 from p in region \mathfrak{S}_3 to a point p' in region \mathfrak{S}_1 . Tracing the path P_1 , C decays across L_C . But along the path P_2 which is homotopic to P_1 , on the other hand, L_B is crossed before L_C , B has decayed before arriving at L_C . This challenges our hypothesis of stability of C in the region \mathfrak{S}_2 . Let us now appeal to the octahedral axiom, (2.9). Given the two bound states $C = A + B$ and $B = E + F$, we have two triangles $A \rightarrow B \rightarrow C \rightarrow A[1]$ and $E \rightarrow F \rightarrow B \rightarrow E[1]$, respectively. Let us rewrite the latter one in the homotopy equivalent form $B \rightarrow E[1] \rightarrow F[1] \rightarrow B[1]$, in order to bring the triangles to the form (2.7). Let us define $\phi_0 = \phi(A)$ and evaluate the grades of other objects using (2.9), with E and F replaced with $E[1]$ and $F[1]$, respectively and assuming that we are on the line L_C in the region $\mathfrak{S}_1 \cup \mathfrak{S}_3$. We write the grades in a tabular form corresponding to the objects in the diagram (2.9):

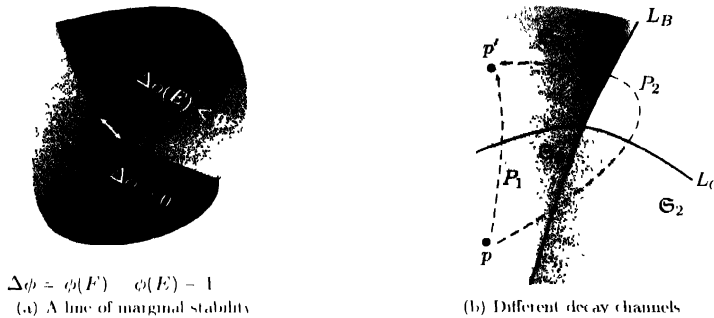


Figure 2 Marginal stability

$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$	ϕ_0	$\phi_0 + 1$	$\phi_0 + 1$	$\phi_0 + 1$
$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$				
$A \longrightarrow E[1] \longrightarrow M \longrightarrow A[1]$	ϕ_0	$\phi_0 + 1$	$\phi_0 + 1 - \varepsilon$	$\phi_0 + 1$
$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$				
$B \longrightarrow E[1] \longrightarrow F[1] \longrightarrow B[1]$	$\phi_0 + 1$	$\phi_0 + 1 - \varepsilon'$	$\phi_0 + 2 + \varepsilon''$	$\phi_0 + 2$
$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$				
$C \longrightarrow M \longrightarrow F[1] \longrightarrow C[1]$	$\phi_0 + 1$	$\phi_0 + 1 - \varepsilon$	$\phi_0 + 2$	$\phi_0 + 2$

(2.13)

where ε , ε' and ε'' are positive numbers. Let us now trace L_C from left to right, by changing the Kahler modulus. C continues to be a bound state, while destabilizing B in the process to decay into $E + F$. That is, we have

$$\phi(F) - \phi(E) - 1 > 0 \quad (2.14)$$

Since we have not altered $\phi(A)$ and hence $\phi(B)$, we derive from the table that in this process $\phi(F)$ increases while $\phi(E)$ and $\phi(M)$ decrease. Then by the second row of the commutative diagram,

$$\phi(E[1]) - \phi(A) = 1 - \varepsilon' < 1, \quad (2.15)$$

$$0 = E_0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \cdots \longrightarrow E_{n-1} \longrightarrow E_n = E$$

$\swarrow [1]$
 A_1

$\swarrow [1]$
 A_2

$\swarrow [1]$
 A_3

$\swarrow [1]$
 A_{n-1}

$\swarrow [1]$
 A_n

(2.16)

then it may decay into A_1, A_2, \dots, A_n , if

$$\phi(A_1) > \phi(A_2) > \cdots > \phi(A_n) \quad (2.17)$$

Let us remark that this is where the Harder-Narasimhan

ergo M is now stable. Thus, while B decays across L_B , the bound state C continues to be a stable state, a bound state of objects different from A and B and decays only across L_C . We thus learn that while the decay of a brane into two daughters is studied using a distinguished triangle, the decay of a brane into three needs invoking the octahedral axiom. In order to consider multiple decays we need to consider a set of distinguished triangles corresponding to a brane E [24, 25]. If E can be written as

property of the slope function (2.11) is put to use. However, as we have discussed above, for the decay to take place, the decay products have to be stable. It is deemed

at each point of the complexified Kähler moduli space \mathcal{M} there exists a set of stable D-branes so that the objects $D^b(\text{Coh } \mathcal{M})$ corresponding to any given brane can be expressed as (2.16) for a certain integer n and a set of objects $\{A_n\}$ satisfying (2.17). The stable objects can be chosen from the heart of a t-structure, which by the fact being an Abelian category is amenable to the notion of an order. Thus, to every point of the Kähler moduli space should be assigned a t-structure in whose heart the given objects A_n take residence. This requires mapping the t-structures of $D^b(\text{Coh } \mathcal{M})$ on the Kähler moduli space of \mathcal{M} . This is a formidable task in general and which is known about the t-structures for Calabi-Yau manifolds. However, the t-structures on $D^b(\text{Coh } \mathbf{P}^1)$ have been classified. In the next section we shall consider an assignment of t-structures to the points in the Kähler moduli space of \mathbf{P}^1 which arises from a geometrically engineered Calabi-Yau.

Before this, let us briefly discuss some features of the objects which will be useful in the calculations. As remarked above, at each point in the Kähler moduli space, a set of basic branes is deemed to exist and any brane in the moduli space is to be made up of these basic ones. The stability of a brane is ascertained with respect to decaying these basic branes. But this surmises the stability of these basic branes a priori. This leads to a circularity in the notion of stability that we have discussed so far. The solution is to assume that there is some other way of bounding the class of stable objects at a certain given point in the moduli space, consistent with Π -stability, and to apply the definition in terms of the grade ϕ as other points of the moduli space is traversed. One convenient criterion for the other criterion is the μ -stability of sheaves in the region in the moduli space. But this necessitates a reidentification of an Abelian category of sheaves in the region. We need to know the different t-structures of the moduli category realized in various regions in the Kähler moduli space. To the large volume limit of \mathcal{M} , which is a region in the Kähler moduli space, where geometric arguments are valid, we then ascribe the Abelian category of \mathcal{M} and employ the orthodox μ -stability of sheaves to determine if a brane is stable in this region. Once this criterion is worked out, we can move along paths penetrating the deep interior of the Kähler moduli space and ask stability of branes.

To be more concrete, let us consider type-IIA theory on $\mathbf{R}^{1,3} \times \mathcal{M}$, where \mathcal{M} is a Calabi-Yau manifold [4, 18, 19, 24, 33]. The topological D-branes in the corresponding B-model are objects of the bounded derived category of coherent sheaves on \mathcal{M} , $D^b(\text{Coh } \mathcal{M})$. For an

object E of $\text{Coh } \mathcal{M}$ the central charge is given by

$$Z(E) = - \int_{\mathcal{M}} e^{-tK} \text{ch}(E) \sqrt{\text{Td}(\mathcal{M})} + \quad (2.18)$$

in the large volume limit, where K denotes the generator of the Kähler cone, t the Kähler parameter, complexified with the anti-symmetric field B which appeared in the Dirac-Born-Infeld action (1.4). $\text{ch}(E)$ denotes the Chern character of E and $\text{Td}(\mathcal{M})$ denotes the Todd class of \mathcal{M} and the integration is over \mathcal{M} . The grade is then given by

$$\phi = -\frac{1}{\pi} \text{Im} \log Z(E) \quad (2.19)$$

In order to study the stability of branes in the Kähler moduli space of \mathcal{M} , then, it turns out to be convenient to fix a base-point in the large volume limit where the basic branes are taken to be the ones residing in $\text{Coh } \mathcal{M}$, which is the heart of the canonical t-structure of $D^b(\text{Coh } \mathcal{M})$, as mentioned before. It is with respect to this base-point that we assign grades to the branes and t-structures at other points in the moduli space.

3 A heart for a stable brane

Let us now discuss an application of the various aspects of Π -stability discussed so far with a simple example. This serves to illustrate the usefulness of the intricate machinery in a simple physical problem. In particular, we describe a limit of a string theory compactification where the BPS states naturally arise as objects of derived category of coherent sheaves on the projective curve \mathbf{P}^1 . We also work out the atlas of t-structures over the Kähler moduli space of \mathbf{P}^1 .

3.1 Degenerating the target

Let us consider a Calabi-Yau threefold which is a **K3** fibration on \mathbf{P}^1 . The compactified theory has a gauge sector with gauge fields corresponding to the open strings connecting D-branes wrapped on various compact homology cycles of the Calabi-Yau. Choosing the volume of the projective curve to be large and that of the **K3** appropriate, we obtain a non-Abelian gauge theory.

More specifically, let us consider the space \mathcal{M} as the degree 8 hypersurface in the resolution of the weighted projective space $\mathbf{P}^4[2, 2, 2, 1, 1]$. In order to study phenomena in the moduli space of the complexified Kähler two-form of \mathcal{M} including non-perturbative influences we have recourse to Mirror symmetry and consider the mirror

dual \mathcal{W} of \mathcal{M} which is given by a $\mathbf{Z}_4 \times \mathbf{Z}_4 \times \mathbf{Z}_4$ quotient of the same hypersurface and is described by the equation

$$a_0 z_1 z_2 z_3 z_4 z_5 + a_1 z_1^4 + a_2 z_2^4 + a_3 z_3^4 + a_4 z_4^8 + a_5 z_5^8 + a_6 z_4^4 z_5^4 = 0, \quad (3.1)$$

where z_1, \dots, z_5 denote the coordinates of the five-dimensional affine complex space \mathbf{C}^5 and the parameters a_i represent complex deformations of the polynomial thus corresponding to deformation of complex structure. The algebraic coordinates of the complex structure moduli space are

$$x = \frac{a_1 a_2 a_3 a_6}{a_0^4}, \quad y = \frac{a_4 a_5}{a_6^2}, \quad (3.2)$$

obtained by rescaling the affine coordinates $\{z_i\}$. The Mirror map relates these coordinates to the complexified Kahler moduli. The two Kahler moduli corresponding to x and y , denoted $(B + iJ)_x$ and $(B + iJ)_y$, represent, respectively, the sizes of $\mathbf{K3}$ and \mathbf{P}^1 . These are obtained as solutions to the Picard-Fuchs equations for the periods and are given by

$$(B + iJ)_x = \frac{1}{2\pi i} \log t + \mathcal{O}(x, y), \quad (3.3)$$

$$(B + iJ)_y = \frac{1}{2\pi i} \log y + \mathcal{O}(x, y),$$

where $\mathcal{O}(x, y)$ represents linear and higher order terms in x and y . The mirror \mathcal{W} is an algebraic variety which is singular along a *discriminant locus*, $\Delta = \Delta_0 \Delta_1^3$, with the *primary* component, Δ_0 , given by

$$(1 - 2^8 x)^2 - 2^{18} x^2 y = 0, \quad (3.4)$$

and Δ_1 given by $1 - 4y = 0$. In the limit in which the volume of the base \mathbf{P}^1 is large, that is $y \rightarrow 0$, the gauge symmetry of the configuration enhances to a non-Abelian one, namely, $SU(2)$. The non-perturbative nature of this enhancement dictates restricting our considerations to the discriminant locus, thereby forcing a choice of $x = 1/2^8$ too. By the double scaling limit of $(x, y) \rightarrow (2^{-8}, 0)$ and setting $u = \frac{1-2^8 x}{2\sqrt{y}}$ we can consider the Kahler moduli space in the neighborhood of the point of enhanced symmetry. This procedure is called *geometric engineering* [35]. The moduli space of interest is now the u -plane, intersecting the principal component Δ_0 at $u = \pm 1$. This can be identified as the u -plane of the Seiberg-Witten $SU(2)$ theory. The supersymmetric states in this theory arise from the branes in \mathcal{M} which become massless at $(x, y) = (2^{-8}, 0)$. We envisage these branes as being objects in the derived category of coherent sheaves on \mathbf{P}^1 [31]. This construction provides a simple physical arena for the derived category picture of branes to be realized. Indeed, for sufficiently

large volumes all the objects in $D^b(\text{Coh } \mathbf{P}^1)$ can be generated by the structure sheaf $\mathcal{O}_{\mathbf{P}^1}$ on the projective curve physically corresponding to a D4-brane wrapped on the $\mathbf{K3}$ and the sky-scraper sheaf \mathcal{O}_x which corresponds to a D6-brane wrapped on the whole of \mathcal{M} , where x is a point in \mathbf{P}^1 .

It thus suffices to consider Π -stability on the derived category of coherent sheaves on \mathbf{P}^1 . The stability condition requires defining a centered slope function which is a map from the K-group of \mathbf{P}^1 to \mathbf{C} . The K-group of \mathbf{P}^1 is generated by $H^0(\mathbf{P}^1)$ and $H^2(\mathbf{P}^1)$. We can take the two generators of $D^b(\text{Coh } \mathbf{P}^1)$ namely $\mathcal{O}_{\mathbf{P}^1}$ and \mathcal{O}_x to be generators of the K-group. The Chern characters of the sheaves span $H^0(\mathbf{P}^1)$ and $H^2(\mathbf{P}^1)$ respectively. Therefore to obtain the slope function of an object we require two basic slopes, determined by the phases of the central charge

$$a = Z(\pi^* \mathcal{O}_x), \quad a_D = Z(\pi^* \mathcal{O}_{\mathbf{P}^1}), \quad (3.5)$$

where a and a_D are given in terms of the periods of homology cycles. Here π^* denotes the pull-back of the $\pi: \mathcal{M} \rightarrow \mathbf{P}^1$, signifying that the branes in \mathcal{M} are obtained from the sheaves on \mathbf{P}^1 . The periods are obtained as solutions to the Picard-Fuchs equation associated with the Calabi-Yau \mathcal{M} , which in the double-scaling limit comes

$$z(1-z) \frac{\partial^2 \Phi}{\partial z^2} - \frac{1}{4} \Phi = 0, \quad (3.6)$$

where we defined $z = (1/2)(u-1)$. The periods are obtained as the two solutions to this equation as

$$a(u) = \sqrt{2(u+1)} {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; \frac{2}{u+1}\right),$$

$$a_D(u) = -\frac{u-1}{2i} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{2}{1-u}\right) \quad (3.7)$$

where the Hypergeometric functions are defined on the u -plane with branch-cuts from $(-\infty, 0)$ for a and $(1, \infty)$ for a_D . Slope functions for an arbitrary sheaf on \mathbf{P}^1 can be obtained from these expressions for the periods. For example, the central charge of $\pi^* \mathcal{O}_{\mathbf{P}^1}(n)$ is given by

$$Z(\mathcal{O}(n)) = a_D(u) + na(u), \quad (3.8)$$

and its phase determines the slope function at any point u .

The simplicity of this example owes its origin mainly to the fact that there is but a single curve of marginal stability, an ellipse, homotopic to the circle $|u| = 1$ and passing through $u = \pm 1$, as shown by the dotted line in Figure 3(a). The stable states outside the ellipse are $\pi^* \mathcal{O}_{\mathbf{P}^1}(n)$, with $n \in \mathbf{Z}$ and $\pi^* \mathcal{O}_x$, with $x \in \mathbf{P}^1$. Inside the circle the stable states are only $\pi^* \mathcal{O}_{\mathbf{P}^1}$ and either $\pi^* \mathcal{O}_{\mathbf{P}^1}(1)$ and $\pi^* \mathcal{O}_{\mathbf{P}^1}(-1)$, depending on the direction entry into the interior. These reproduce the the totality

ies in the dyon spectrum of the Seiberg-Witten theory
th in the weak and the strong coupling regimes, which
e identified as the regions outside and inside of the el-
se respectively. The string junctions and spiral strings
the moduli space have also been identified among these
ple objects [30, 32].

2 Hearts with perverse sheaves

re t-structures bring out the Abelian categories within
derived category. T-structures on the derived cat-
ory $D^b(\text{Coh } \mathbf{P}^1)$ have been classified [34] upto au-
equivalences of the derived category, forming a group
d, $D^b(\text{Coh } \mathbf{P}^1)$). As a preparation to constructing the
as of t-structures on the moduli space let us first take
ack of the bounded t-structures on $D^b(\text{Coh } \mathbf{P}^1)$. There
two classes of t-structures on $D^b(\text{Coh } \mathbf{P}^1)$, namely,
standard and exceptional. We begin with the classifica-
on of the standard t-structures.

The simplest t-structure on $D^b(\text{Coh } \mathbf{P}^1)$ is the tauto-
logical one with $\text{Coh } \mathbf{P}^1$ as its heart, given by

$$\begin{aligned}\mathcal{A}^{\leq 0} &= \langle \text{Coh } \mathbf{P}^1[j], j \geq 0 \rangle, \\ \mathcal{A}^{\geq 0} &= \langle \text{Coh } \mathbf{P}^1[j], j \leq 0 \rangle.\end{aligned}$$

a heart is $\mathcal{A}^\heartsuit = \text{Coh } \mathbf{P}^1$ which contains only branes and
antibranes. This is the t-structure that we posit to be
sized in the large volume limit.

Other standard t-structures are obtained from $\text{Coh } \mathbf{P}^1$
constructing cotilting torsion pairs. We shall not delve
in the details of the construction. For our purposes it
suffices to recall that whenever an Abelian category \mathcal{A}
can be split into a torsion pair $(\mathcal{A}_1, \mathcal{A}_0)$ of full subcate-
gories satisfying certain criteria [22, 34], we can obtain a
structure from it as

$$\begin{aligned}{}^p D^b(\mathcal{A})^{\leq 0} &= \{A \in D^b(\mathcal{A})^{\leq 0} | H^0(A) \in \mathcal{A}_1\}, \\ {}^p D^b(\mathcal{A})^{\geq 0} &= \{A \in D^b(\mathcal{A})^{\geq -1} | H^{-1}(A) \in \mathcal{A}_0\},\end{aligned}$$

whose heart is the category of p -perverse sheaves on \mathbf{P}^1 .
2: We can thus start from the Abelian category $\text{Coh } \mathbf{P}^1$
named as the heart of the tautological t-structure and
define others with perverse sheaves through cotilting. The
st torsion pair that is constructed in this vein is

$$\begin{aligned}\mathcal{B}_0 &= \langle \mathcal{O}(n), n < 0 \rangle, \\ \mathcal{B}_1 &= \langle \mathcal{O}(n), n \geq 0; \mathcal{O}_x, x \in \mathbf{P}^1 \rangle\end{aligned}$$

The associated t-structure, then, is given by

$$\begin{aligned}\mathcal{B}^{\leq 0} &= \langle \mathcal{O}(n)[i], n \geq 0, i \geq 0, \mathcal{O}_x[i], x \in \mathbf{P}^1, i \geq 0; \\ &\quad \mathcal{O}(n)[j], n < 0, j \geq 1 \rangle,\end{aligned}$$

where we have recorded only one half of the t-structure
since a t-structure is unambiguously specified by a moi-
ety. The heart of the t-structure is

$$\mathcal{B}^\heartsuit = \langle \mathcal{O}(n)[0], n \geq 0; \mathcal{O}_x[0], x \in \mathbf{P}^1; \mathcal{O}(n)[1], n < 0 \rangle \quad (3.10)$$

The heart is different from that of the tautological t-
structure. In particular, let us remark that the heart \mathcal{B}^\heartsuit
contains objects which would be interpreted as antibranes
in the parlance of the large volume regime.

There are two other cotilting torsion pairs that
completes the list of standard bounded t-structures on
 $D^b(\text{Coh } \mathbf{P}^1)$. One of these is

$$\mathcal{C}_0 = \langle \mathcal{O}(n), n \in \mathbf{Z} \rangle, \quad \mathcal{C}_1 = \langle \mathcal{O}_x, x \in \mathbf{P}^1 \rangle,$$

giving rise to the t-structure

$$\mathcal{C}^{\leq 0} = \langle \mathcal{O}(n)[i], n \in \mathbf{Z}, i \geq 0, \mathcal{O}_x[i], x \in \mathbf{P}^1, i \geq 0; \rangle,$$

with the heart

$$\mathcal{C}^\heartsuit = \langle \mathcal{O}(n)[1], n \in \mathbf{Z}; \mathcal{O}_x[0], x \in \mathbf{P}^1 \rangle \quad (3.11)$$

The other torsion requires an arbitrary nonempty subset
 $P \subset \mathbf{P}^1$ [34]. This t-structure is not realized on the moduli
space. So we refrain from listing it.

There are two kinds of exceptional t-structures:
bounded and *unbounded*. As mentioned earlier, we are
interested only in the bounded t-structures of which there
are but two in the derived category $D^b(\text{Coh } \mathbf{P}^1)$. These
depend on an integer $k \in \mathbf{Z}$, and are given by

$$\begin{aligned}\mathcal{E}^{\leq 0} &= \langle \mathcal{O}[i], i \geq k, \mathcal{O}(1)[j], j \geq -2, \\ \mathcal{F}^{\leq 0} &= \langle \mathcal{O}[i], i \geq k, \mathcal{O}(1)[j], j \geq -1,\end{aligned}$$

with hearts

$$\mathcal{E}^\heartsuit = \langle \mathcal{O}(k)[0], \mathcal{O}(1)[-2] \rangle, \quad (3.12)$$

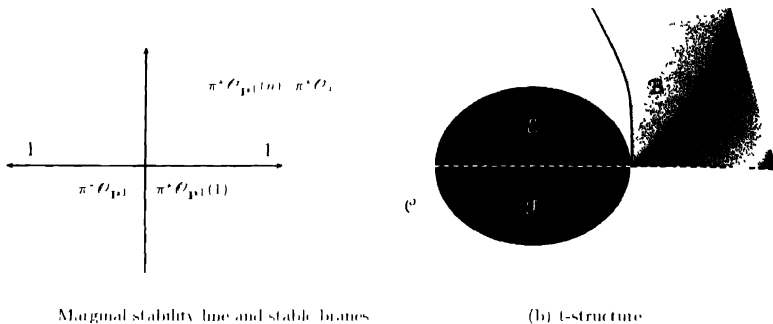
$$\mathcal{F}^\heartsuit = \langle \mathcal{O}(k)[0], \mathcal{O}(1)[-1] \rangle, \quad (3.13)$$

respectively.

in the slope function then the various t-structures make
their appearances as we wander about the moduli space.
As discussed above, to obtain an atlas of t-structures in

3 Branes in the hearts

nally, let us discuss the variation of t-structures over the
moduli space of Kahler volume of \mathbf{P}^1 . If we fix a window

Figure 3: Kahler moduli space of \mathbf{P}^1

the moduli space the slope function is to be calibrated by matching with the μ -stability criterion in the large volume limit. In this region stable objects are given by the invertible sheaves or line bundles on \mathbf{P}^1 . Since the heart is expected to consist of stable objects only, it has to contain $\mathcal{O}(n)$, for integers, n and the t-structure corresponds to some standard t-structure. On the other hand, within the confines of the line of marginal stability it is only \mathcal{O} and $\mathcal{O}(1)$ or $\mathcal{O}(-1)$, which are stable. So the candidate t-structure in this region would be exceptional t-structure. Let us mention a point about notation in the following. We shall consider sheaves on \mathbf{P}^1 from now on. To interpret these as branes we have to pull-back the sheaves on \mathcal{M} with π^* . We shall suppress both the \mathbf{P}^1 and the pull-back from the notation in the following for typographical ease. Let us now chart out the atlas of t-structures on the Kahler moduli space of \mathbf{P}^1 .

- First, let us consider the region in the u -plane with $\text{Re } u > 1$ and $\text{Im } u$ small and positive. This includes the large radius region. Looking for objects whose phases lies in the interval $\phi \in [-1, 0]$, we find the set of objects, $\langle \mathcal{O}(n), n \in \mathbf{Z}, \mathcal{O}_x, x \in \mathbf{P}^1 \rangle$. This set consists of branes only, with no anti-branes. We identify this with the heart of the tautological t-structure, \mathcal{A}^\vee . In other words, we assign the t-structure \mathcal{A} to this region of the moduli space. This is marked in Figure 3(b).
- Let us now move away from the positive $\text{Re } u$ -axis counterclockwise. Grades of some of the objects, namely, $\mathcal{O}(-n)$, with $n > N$ for some integer N , will fall below -1 , outside the grade window. While these objects are defenestrated, their shifted cousins

$\mathcal{O}(-n)[1]$ make an entry though the other $\mathcal{O}(n)$ of the window. At a generic point the set of objects carrying grades within the window consists $\langle \mathcal{O}(n), n \in \mathbf{Z}, n > -N, \mathcal{O}(n)[1], n < -N, \mathcal{O}_x, x \in \mathbf{P}^1 \rangle$. Apparently, these do not have a place in the heart of any of the t-structures listed above. But we recall that t-structures are classified upto autoequivalences of the category. One such autoequivalence can be generated by a monodromy transformation associated with the path followed in our journey around the large volume point which transform sheaf by tensoring it with $\mathcal{O}(1)$, which results in change in the grade. Transforming the objects the monodromy N times we can coerce the object to be in the heart of the t-structure \mathcal{B} , as marked in Figure 3(b). These will realize the spiral structure identified earlier [32].

- Continuing our journey in the u -plane, we find that eventually all the $\mathcal{O}(n)$ are replaced by the shifted cousins and beyond a certain line, shown in Figure 3(b), it is only these cousins $\langle \mathcal{O}(n)[1], n \in \mathbf{Z}, \mathcal{O}_x, x \in \mathbf{P}^1 \rangle$ that find a place in the heart of t-structure \mathcal{C} . The t-structure \mathcal{C} is ascribed to white space in Figure 3(b).
- Continuing further around the ellipse as we step stable upon the positive $\text{Re } u$ -axis from below, due to presence of branch cut introduced for defining periods, the grades jump and we come back to regime of the t-structure \mathcal{A} .
- The above discussion is valid outside the elliptical marginal stability. As long as we do not cross the line of marginal stability, the standard t-structure

rough. The branes incarcerated within the con-
 junctures of the marginal stability locus, on the other
 hand, correspond to the exceptional t-structures.
 In the region $\text{Im } u < 1$, the branes take resi-
 dence in the heart of the t-structure \mathcal{E} consisting
 of $\langle \mathcal{O}, \mathcal{O}(1)[-2] \rangle$.

-) Below the $\text{Re } u$ -axis the phase of $a_D(u)$ jumps across
 the branch cut. The basic branes now find an abode
 in the heart \mathcal{F} of the t-structure \mathcal{F} , with the basic
 branes $\langle \mathcal{O}, \mathcal{O}(1)[-1] \rangle$.

Epilogue

I have reviewed some aspects of topological branes in
 type IIA theory which are described by the B-model.
 These geometric notions turn out to be inadequate
 to describe the D-branes precisely. Branes wrapped on
 homology cycles of a Calabi-Yau manifold when the
 IIA theory is compactified on it are geometrically
 viewed as objects in the derived category of coherent
 sheaves on the Calabi-Yau. We recalled several features
 of the derived category before discussing the definition of
 stability of objects and some of its subtleties. It is ex-
 pected that to each point in the Kähler moduli space of
 Calabi-Yau is associated a t-structure in whose heart
 these constituent D-branes reside. We then considered
 a simple example of a geometrically engineered Calabi-
 Yau with only the projective curve \mathbf{P}^1 as its non-trivial
 cycle giving rise to the $SU(2)$ Seiberg-Witten gauge the-
 ory. The dyon spectrum of the gauge theory is reproduced
 in the parlance of the derived category in an elegant
 way. Finally, we associated some of the t-structures of
 the derived category of coherent sheaves on \mathbf{P}^1 to different
 regions in the Kähler moduli space.

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